Algebraic sets

Let k be a field. For our purposes, we only want to consider fields that are algebraically closed. (We will see why soon!)

Def: A field k is algebraically closed if every polynomial with coefficients in k has a root in k. That is, if $f \in k[x]$, there is some $\alpha \in k$ s.t. $f(\alpha) = 0$.

Note: By induction, this implies that every polynomial in $k[x]$ is the product of linear factors. i.e $f = \beta(x-\alpha_1)\dots(x-\alpha_n)$ for some $\beta, \alpha_1, \dots, \alpha_n \in k$.

Ex: $\mathbb{C}$ is algebraically closed (see Math 123 :)), but x^2+1 has no roots in $\mathbb{R}$ so $\mathbb{R}$ is not algebraically closed.

Def: Affine n-space, denoted $\mathbb{A}_k^n$, or just $\mathbb{A}^n$, is the set of n-tuples of elements of k (to distinguish it from the vector space k^n).

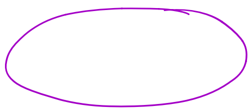
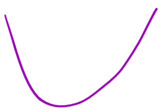
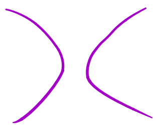
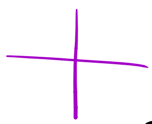
Let $f \in k[x_1, \dots, x_n]$. Then $(a_1, \dots, a_n) \in \mathbb{A}^n$ is a zero of f if $f(a_1, \dots, a_n) = 0$

Ex: Conics in $\mathbb{A}_{\mathbb{C}}^2$.

A conic is the set of zeros of a quadratic equation:

$$g(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$$

If we just consider the real locus, we get the following familiar conics:

- ellipse 
- parabola 
- hyperbola 
- two lines (e.g. xy) 
  if g is reducible over $\mathbb{R}$

But the $\mathbb{R}$ locus doesn't always give a very complete picture:

In $\mathbb{R}^2$: x^2+y^2 defines a single point, and x^2+y^2+1 defines the empty set, whereas they have infinitely many solutions over $\mathbb{C}$.

Note: It usually makes sense to require $a, b,$ or c to be nonzero to avoid lines (e.g. x) and the whole plane (0).

More generally, we can describe zero loci using more than one polynomial:

Def: Let $S \subseteq k[x_1, \ldots, x_n]$ be a set of polynomials.

Define $V(S) := \{ P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in S \}$

$X \subseteq \mathbb{A}^n$ is an <u>algebraic set</u> if $X = V(S)$ for some S.

Ideals

Let $S \subseteq k[x_1, \ldots, x_n]$, and let I be the ideal generated by S.

Claim: $V(S) = V(I)$.

Pf: If $P \in V(I)$, then $f(P) = 0$ $\forall f \in S$ since $S \subseteq I$. Thus, $P \in V(S)$.

If $P \in V(S)$, then if $g \in I$, $g = a_1 f_1 + \ldots + a_m f_m$ for some $a_i \in k[x_1, \ldots, x_n]$, $f_i \in S$. Thus $g(P) = 0 + \ldots + 0 = 0$, so $P \in V(I)$. $\square$

Cor: Every algebraic set is equal to $V(I)$ for some ideal I.

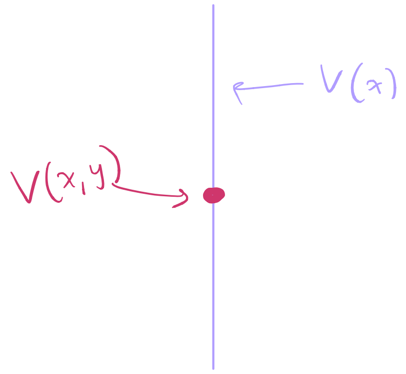
Ex: If $f \in k[x_1, \ldots, x_n]$, and $I = (f)$, then $V(f) = V(I)$. In this case, $V(I)$ is called a hypersurface.

If $f = y \in k[x,y]$, $V(f) =$ The x-axis.

We can now deduce several basic properties of $V(S)$:

1.) It's inclusion-reversing: If $I \subseteq J$ then $V(I) \supseteq V(J)$.

<u>Ex</u>: $(x) \subseteq (x,y)$



2.) If $\{I_\alpha\}$ is a collection of ideals,

$$\bigcap_\alpha V(I_\alpha) = V\Big(\bigcup_\alpha I_\alpha\Big) = V\big(\text{ideal gen. by } I_\alpha\big)$$

(i.e. intersections of algebraic sets are alg. sets.)

3.) $f, g \in k[x_1, \ldots, x_n] \Rightarrow V(f) \cup V(g) = V(fg)$.

More generally, $V(I) \cup V(J) = V(IJ)$

(i.e. finite unions of alg. sets are alg. sets).

[<u>Note</u>: Infinite (even countable!) unions are not always alg. sets: If $X = V(I) \subseteq \mathbb{A}^1_{\mathbb{C}}$, then $f \in I$ is 0 or has finitely many roots, so X is finite or all of $\mathbb{A}^1$.]

4.) $V(0) = \mathbb{A}^n$, $V(1) = \emptyset$, and $V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\}$

so any finite set is algebraic.

Properties 2.) - 4.) show that algebraic sets behave like closed sets. In fact...

Def: $X \subseteq \mathbb{A}^n$ is a Zariski closed set if X is an algebraic set. Y is Zariski open if $\mathbb{A}^n \setminus Y$ is Zariski closed.

The collection of Zariski open sets in $\mathbb{A}^n$ is called the Zariski topology.

Ex: The Zariski open sets on $\mathbb{A}^1$ are the empty set and the cofinite sets.

Note: If you know any topology, the Zariski topology is strictly coarser that the standard Euclidean topology. i.e. algebraic sets are all closed in the standard topology as well.